THE STEINBERG REPRESENTATION OF $GL_3(\mathbb{F}_2)$

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1. Introduction

Let V be a vector space over $\mathbb{F}_2$ of dimension 3, and consider its automorphism group $G = GL(V) \cong GL_3(\mathbb{F}_2)$. This is a finite simple group of order $|G| = 168$. To see this, note a 3×3 matrix A is invertible iff its columns are linearly independent. To choose three such vectors, we must first choose a nonzero vector a, followed by a nonzero vector b which is not a, followed by a nonzero vector c which is neither a nor b and not in their span $a + b$. This yields $(2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 7 \cdot 6 \cdot 4 = 168$ possibilities.

Note that since G is a finite group, we can compute its character table by considering irreducible representations over $\mathbb{C}$. It turns out these have dimension 1, 3, 3, 6, 7, 8. We can verify the dimension formula $1^2 + 3^2 + 3^2 + 6^2 + 7^2 + 8^2 = 1 + 9 + 9 + 36 + 49 + 64 = 168$. We will briefly sketch the construction of smaller representations, with our goal being to explicitly construct the largest eight dimensional representation, the *Steinberg representation*.

- d = 1: trivial rep'n
- d = 3: standard rep'n
- d = 3: dual to standard rep'n
- d = 6: Λ^2(std. rep'n)
- d = 7: action on $\mathbb{P}^1(\mathbb{F}_7)$ via $PSL_2(\mathbb{F}_7)$ isomorphism minus trivial rep'n
- d = 8: Steinberg representation

The full character table is given in Table 1.

2. Tits Building

In order to construct the Steinberg representation, we will explicitly describe the action of G on $H_1(\Delta(G); \mathbb{C})$, where $\Delta(G)$ is the *Tits building* associated to G, a simplicial complex whose simplices correspond to flags of proper nontrivial subspaces of the vector space $V \cong \mathbb{F}_2^3$.

To construct the building, note that a nontrivial flag is given by $p \subset H$ where $p \in V$ is a point corresponding to the 1 dim'l line $\text{span}_{\mathbb{F}_2}\{p\} := \langle p \rangle$, and H is a 2 dim'l plane corresponding to two basis vectors $\langle v_i, v_j \rangle$. We can also describe planes H as the vanishing set

Date: January 3, 2026.

Class	1	2	3	4	$7A$	$7B$	
Size	1	21	56	42	24	24	
ρ_1	1	1	1	1	1	1	trivial
ρ_2	3	-1	0	1	$\frac{-1+\sqrt{-7}}{2}$	$\frac{-1-\sqrt{-7}}{2}$	complex faithful
ρ_3	3	-1	0	1	$\frac{-1-\sqrt{-7}}{2}$	$\frac{-1+\sqrt{-7}}{2}$	complex faithful
ρ_4	6	2	0	0	-1	-1	orthogonal faithful
ρ_5	7	-1	1	-1	0	0	orthogonal faithful
ρ_6	8	0	-1	0	1	1	orthogonal faithful

TABLE 1. Character table of G with conjugacy classes, sizes, and irreducible characters

of a linear functional $H_v := \{x : x \cdot v = 0\}$, e.g. $H_{(1,0,0)} = \{x : (1,0,0) \cdot x = 0\} = \{x_1 = 0\}$. There are 7 points and 7 planes in V. There are $3 \cdot 7 = 21$ incidence relations corresponding to the incidence graph of the Fano plane where we have an edge if $p \subset H$. Thus we obtain an oriented, bipartite, trivalent graph with orientation given by the boundary map $\partial(p \subset H) = H - p$.

For concision, let p_i denote the point with coordinates given by the binary representation of i, e.g. $i = 5 \leftrightarrow 101 \leftrightarrow (1,0,1)$. That is, if $i = b_0 2^0 + b_1 2^1 + b_2 2^2$ in binary, then $p_i = (b_0, b_1, b_2)$. Let H_i denote the plane given by the linear equation $\{x \cdot p_i = 0\}$, e.g. $H_5 = \{x \cdot p_5 = 0\} = \{x_0 + x_2 = 0\}$.

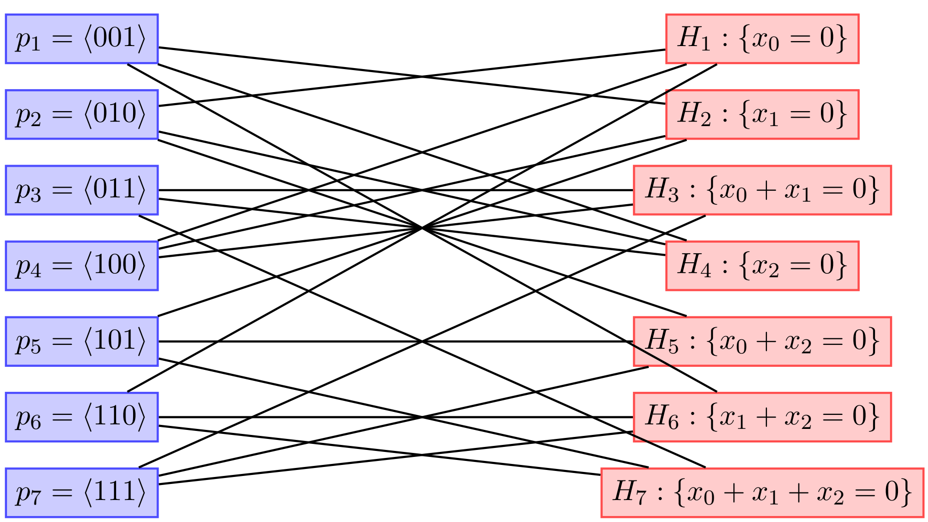

FIGURE 1. The Tits building $\Delta(G)$ of $GL_3(\mathbb{F}_2)$ as a bipartite incidence graph.

A *chamber* is a maximal simplex in the building, so in this case would correspond to a maximal flag. There are $3 \times 7 = 21$ such flags which are just the edges of the incidence graph.

An *apartment* is a subcomplex of Δ which is isomorphic to the Coxeter complex of the Weyl group W.

The *Weyl group* of a reductive algebraic group, or in this case a finite group of Lie type, is $W := N_G(T)/T$ where $T \subset G$ is a maximal torus. In this case, the maximal torus consists of diagonal matrices, so $T \cong \left(F_2^\times\right)^3 \cong \{Id\}$. The normalizer can be written as a semidirect product of diagonal matrices and permutation matrices (*monomial matrices*), so $N_G(T) = T\Sigma_3$, where Σ_3 are permutation matrices of degree 3. Therefore we see $W = T\Sigma_3/T = \Sigma_3 \cong S_3$.

Each apartment is generated by a choice of basis for V. For example, consider the standard basis $\{e_0, e_1, e_2\}$. We have three lines $\langle e_i \rangle$ and three planes $\langle e_i, e_j \rangle$. We have corresponding edges, e.g. $\langle e_0 \rangle \subset \langle e_0, e_1 \rangle$. In total there are six edges, and we get the hexagon shape in Figure 2.

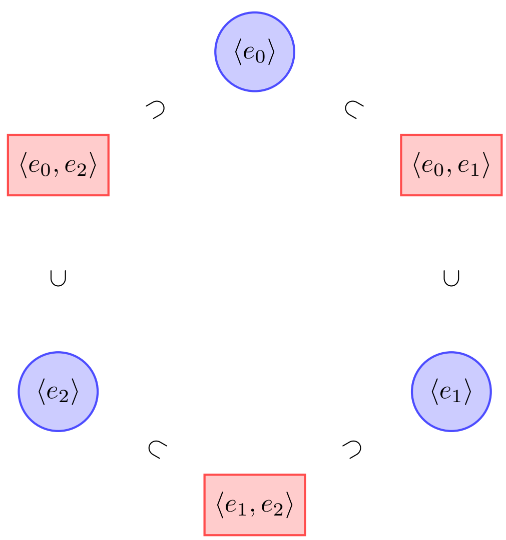

FIGURE 2. The apartment corresponding to the standard basis $\{e_1, e_2, e_3\}$ of V.

The Weyl group W acts on apartments. Let $\sigma \in W \cong S_3$ be a permutation. Then $\sigma \cdot \langle e_i \rangle = \langle \sigma e_i \rangle = \langle e_{\sigma(i)} \rangle$ for lines. Similarly, $\sigma \cdot \langle e_i, e_j \rangle = \langle e_{\sigma(i)}, e_{\sigma(j)} \rangle$ for planes. For example, let $\sigma = (0\,1)$ be the transposition swapping zero and one. Then we obtain the permuted apartment in Figure 3.

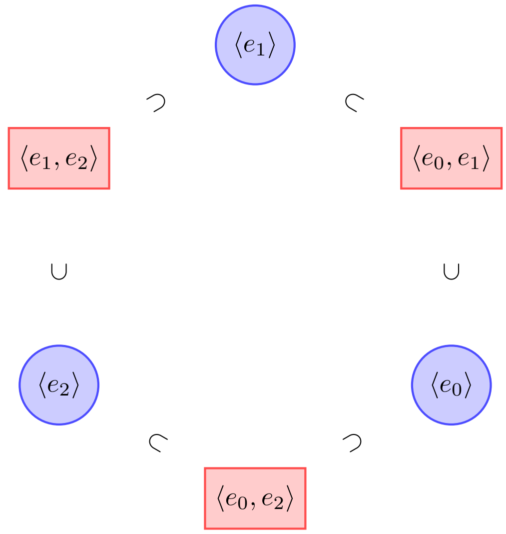

FIGURE 3. The apartment corresponding to the standard basis of V permuted by the transposition $(0\,1)$.

3. COMPUTING $\tilde{H}_1(\Delta(G); \mathbb{C})$

Note that the building is a rank-2 simplicial complex, so we can form the chain complex $0 \to C_1 \xrightarrow{\partial} C_0 \to 0$ where the boundary map again is $\partial[p \subset H] = H - p$ for an oriented edge $p \subset H$ in the incidence graph. The 1-chains are given by formal $\mathbb{C}-$linear combinations of edges, $C_1 = \mathbb{C}[edges]$, and the 0-chains are $C_0 = \mathbb{C}[vertices]$. Note that $\dim(C_1) = 21$ and $\dim(C_0) = 14$ since there are 21 edges and 7 lines + 7 planes.

Note that $H_0(\Delta(G); \mathbb{C}) = \ker(C_0 \to 0)/\mathrm{im}(\partial) \cong C_0/\mathrm{im}(\partial) \cong \mathbb{C}$ since $\Delta(G)$ is a connected graph. This means that $\dim(\mathrm{im}(G)) = 13$, and so we conclude that $\dim(\ker(\partial)) = 21 - 13 = 8$. Therefore $H_1(\Delta(G); \mathbb{C}) = \ker(\partial)/\mathrm{im}(0 \to C_1) \cong \ker(\partial)$. This is the right dimension for the Steinberg representation.

Now, let's compute a basis for $H_1(\Delta(G); \mathbb{C})$. We can represent ∂ as a 14×21 matrix:

$$\begin{bmatrix}
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$$

The kernel of this matrix is an eight dimensional subspace of the row space:

$$\begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}$$

For example, we could look at the first basis vector:

$$(1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 0, 0, 1, 0, -1)$$

Recall that the columns are indexed by edges in the incidence graph, so we really get a signed formal linear combination of six edges:

$$\begin{aligned}
&+ \big[(1,0,0) \subset \{x_3 = 0\}\big] \\
&- \big[(1,0,0) \subset \{x_2 = 0\}\big] \\
&- \big[(0,1,0) \subset \{x_3 = 0\}\big] \\
&+ \big[(0,1,0) \subset \{x_1 = 0\}\big] \\
&+ \big[(0,0,1) \subset \{x_2 = 0\}\big] \\
&- \big[(0,0,1) \subset \{x_1 = 0\}\big]
\end{aligned}$$

Note that for any apartment, the cycle formed from the oriented linear combination of edges will be in $\ker(\partial)$. For example, consider the apartment A we constructed above corresponding to the identity of the group, i.e. the standard basis $\{e_0, e_1, e_2\}$. Let $v_A \in C_1$ be the chain associated to A:

$$v_A = (1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 0, 0, 1, 0, -1)$$

Then we can expand v_A as:

$$\begin{aligned}
&+ \big[(1,0,0) \subset \{x3 = 0\}\big] \\
&- \big[(1,0,0) \subset \{x2 = 0\}\big] \\
&- \big[(0,1,0) \subset \{x3 = 0\}\big] \\
&+ \big[(0,1,0) \subset \{x1 = 0\}\big] \\
&+ \big[(0,0,1) \subset \{x2 = 0\}\big] \\
&- \big[(0,0,1) \subset \{x1 = 0\}\big]
\end{aligned}$$

We can verify $v_A \in \ker(\partial)$ by either applying ∂ directly or multiplying by the matrix $[\partial]$. Let's verify directly:

$$\partial(v_A) = \begin{matrix}
+\,\partial\big[(1,0,0) \subset \{x3 = 0\}\big] \\
-\,\partial\big[(1,0,0) \subset \{x2 = 0\}\big] \\
-\,\partial\big[(0,1,0) \subset \{x3 = 0\}\big] \\
+\,\partial\big[(0,1,0) \subset \{x1 = 0\}\big] \\
+\,\partial\big[(0,0,1) \subset \{x2 = 0\}\big] \\
-\,\partial\big[(0,0,1) \subset \{x1 = 0\}\big]
\end{matrix} = \begin{matrix}
+\{x3 = 0\} - (1,0,0) \\
-\{x2 = 0\} + (1,0,0) \\
-\{x3 = 0\} + (0,1,0) \\
+\{x1 = 0\} - (0,1,0) \\
+\{x2 = 0\} - (0,0,1) \\
-\{x1 = 0\} + (0,0,1)
\end{matrix} = 0$$

Each element $g \in G$ gives rise to an apartment, and each $v_g \in \ker(\partial)$. Up to orientation, there are 28 total unique apartments. The Weyl group W acts on the apartments, and they break into 7 orbits of size $[1, 3, 3, 3, 6, 6, 6]$.

For each $g \in G$, we also have an action on the edge set given by $g \cdot [p \subset H] = [gp \subset gH]$. This action on C_1 is therefore an action on $H_1(\Delta(G); \mathbb{C}) = \ker(\partial) \subset C_1$.

We have now constructed an 8 dim'l vector space over $\mathbb{C}$ with a G-action. It remains to show that this representation is irreducible. We proceed by brute force calculation. Note that for the standard apartment A above, $\dim(\mathrm{span}_{\mathbb{C}}\{Gv_A\}) = \dim(\ker(\partial)) = 8$, and in fact $\mathrm{span}_{\mathbb{C}}\{Gv_A\} = \ker(\partial)$. Since the representation is the G-orbit of a single vector, it is irreducible. ∎